Space-Time Curvature Singularities in Classical Cosmology Due to Gravitational Collapse

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Abstract

This article takes an attempt to analyze space-time singularity within the black hole region. If a star is weightier than the multiple times of mass of the sun, it might suffer a boundless gravitational collapse without attaining any stationary state. The ultimate result of gravitational collapse of a heavier star must essentially be a black hole. Consequently, space-time singularity must be secreted within the black hole area and pivotal message from the singularity cannot be reached to an observer who is waiting outside of black hole or may stay at infinity. This situation faces when the star has finished its internal nuclear fuel that is used to support the external pressure against the interior dragging gravitational forces. The Schwarzschild metric represents a static and stationary exact solution of the Einstein's field equation. It has two singularities at $r = 0$ *and at* $r = 2m$*, where* $r = 0$ *is a true physical singularity, and the event horizon* $r = 2m$ *is an artificial singularity. The coordinate singularity at* $r = 2m$ *can be removed by the use of Kruskal-Szekeres extension, and the genuine space-time singularity at* $r = 0$ *which is concealed within the event horizon at* $r < 2m$ *cannot be removed anyway. Again in Friedmann, Robertson-Walker (FRW) model there presents an unavoidable curvature singularity at* $t = 0$ *which cannot be removed by any coordinate conversion. At this situation, the scale factor* $S(t)$ *also disappears and all materials are crumpled to null size owing to endless gravitational tidal force. In this paper an effort has been arranged to discuss the curvature space-time singularities in some details.*

Key Words: Space-time Singularity; Coordinate Singularity; Kruskal-Szekeres Extension; Big Bang; Black Hole.

1. Introduction

Space-time singularity is a point that lies at the center of a black hole where gravitational forces become very large, density becomes infinite, the volume of the object becomes zero, and space-time entity destroys. In singularity, the geodesics become incomplete. As a result, physical equations work no more to predict what has happened in the singularity (Clarke, 1993; Mohajan, 2013a).

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Einstein's field equation in the presence of gravitating mass can be presented as (Mohajan, 2013a);

$$R_{ij}-\frac{1}{2}g_{ij}R=-\frac{8\pi G}{c^4}T^{ij} \tag{1.1}$$

where $c=3\times10^8$ m/s is the velocity of light and $G=6.673\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$ is the Newtonian gravitational constant. Here T^{ij} is the energy-momentum tensor, R_{ij} is Ricci tensor, and R is Ricci scalar. In the absence of matter, i.e., for empty space, $T_{ij}=T=0$, hence (1.1) becomes,

$$R_{ij}=0\,, \tag{1.2}$$

Equation (1.2) is called Einstein's law of gravitation in matter free world.

The Schwarzschild space-time metric (discuss later) is considered as one of the most studied nontrivial asymptotically flat solution to the Einstein equation (1.2). It is derived from the assumption that it does not change with time, and it is spherically symmetric. It plays a fundamental role in explaining event horizons and space-time singularities. It is significant for the explanation of a non-rotating black hole. A black hole is a very dense region which strongly bends the space-time around it due to its strong gravitation, and nothing can escape from it. The Schwarzschild metric has a difficulty that it dependents on the metric tensor g_{ij} (Mohajan, 2013b).

If the mass of a star surpasses Chandrasekhar limit (discuss later); the star has run short its nuclear fuel then it must experience a boundless gravitational breakdown without receiving any stationary state, and finally it creates a black hole. Consequently, a space-time physical singularity is created (Mohajan, 2013c). If a star is heavier than five times of the mass of the sun, it must finish all its nuclear fuel; as a result, achievement of stationary state is impossible for it. Of course, if it can capable to throw away most of its materials by any method during this evolution, that is supported by supernova explosion (Blumenthal et al., 1984).

Again, in FRW model, the Einstein equation (1.1) implies that, $\varepsilon+3P>0$ always, where P is the pressure and ε is the total density. Then it may create a space-time singularity at $t=0$, since $S^2(t)\to0$ at $t\to0$, as the curvature scalar, $\hat{R}=R^{ij}R_{ij}$ bows to infinity, where $t=0$ is considered as the start of the universe. Hence, in FRW model, a crucial curvature singularity is present at

$t=0$ that cannot be removed completely by any coordinate alteration. At this situation, the scale factor $S(t)$ also disappears and all substances are wrinkled to null volume for endless gravitational tidal force (Hawking & Ellis, 1973).

2. Schwarzschild Singularity

Schwarzschild metric is formed by considering a star that is far away from all the gravitating bodies. It represents the geometry external to a spherically symmetric massive body, for example, a star. The Schwarzschild metric represents a static and stationary exact solution of the Einstein's field equation (Mohajan, 2013b). The distance between two infinitesimally separated points x^i and $x^i + dx^i$ in the space-time is defined as;

$$ds^2 = g_{ij} dx^i dx^j \tag{2.1}$$

where det $(g_{ij}(x)) \neq 0$, $g_{ij} = g_{ji}$.

The static spherically symmetric metric is;

$$ds^2 = x(r)dr^2 + r^2 d\sigma^2 - y(r)c^2 dt^2 \tag{2.2}$$

where $g_{11} = x$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$, $g_{44} = -yc^2$, the determinant, $g = -xyc^2 r^4 \sin^2\theta$ and $g_{ij} = 0$ if $i \neq j$. In (2.2) we have used $d\sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

Now we can write the Ricci tensors as,

$$\begin{aligned}
R_{11} &= \frac{y''}{2y} - \frac{y'^2}{4y^2} - \frac{xy'}{4xy} - \frac{x'}{xr} \\
R_{22} &= \frac{ry'}{2xy} - \frac{rx'}{2x^2} + \frac{1}{x} - 1 \\
R_{33} &= R_{22}\sin^2\theta \\
R_{44} &= c^2\left(-\frac{y''}{2x} + \frac{y'^2}{4xy} + \frac{x'y'}{4x^2} - \frac{y'}{xr} \right).
\end{aligned} \tag{2.3}$$

From (1.2) we $R_{ij} = 0$ and for $i = j$, i.e., in the case of Einstein empty space, we can write from (2.3);

$$\frac{y''}{2y} - \frac{y'^2}{4y^2} - \frac{xy'}{4xy} - \frac{x'}{xr} = 0$$

$$\frac{ry'}{2xy}-\frac{rx'}{2x^2}+\frac{1}{x}-1=0$$
$$-\frac{y''}{2x}+\frac{y'^2}{4xy}+\frac{x'y'}{4x^2}-\frac{y'}{xr}=0\,. \tag{2.4}$$

After a state forward calculation of (2.4) we find $y=1-\frac{2m}{r}$ and $x=\left(1-\frac{2m}{r}\right)^{-1}$. From (2.2) in (t,r,θ,ϕ) coordinates, the Schwarzschild metric for gravitating mass m can be written as (Schwarzschild, 1916);

$$ds^2=-hdt^2+h^{-1}dr^2+r^2d\sigma^2 \tag{2.5}$$

where $h=1-\frac{2m}{r}$, and $d\sigma^2=d\theta^2+\sin^2\theta\,d\phi^2$ is the metric of a unit 2-sphere of the area $A=4\pi\,r^2$ with space-time signature (–1, +1, +1, +1). Here $r=2m$ is the Schwarzschild radius, which is considered an event horizon of a black hole, and $m=\frac{GM}{c^2}$, where $G=6.673\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$, is the Newtonian gravitational constant, $c=10^8\,\mathrm{ms^{-1}}$ is the velocity of light, and M is the point mass at the origin which gives rise to the Newtonian gravitational potential ϕ. The range of timelike coordinate t is $-\infty<t<\infty$, and that of other spacelike coordinates are $0<r<\infty,\ 0<\theta<\pi,\ 0<\phi<2\pi$. At $r\to\infty$ the gravitating mass, $m=0$, and then (2.5) represents the Minkowski metric of flat space-time. The coordinate r is restricted by the condition $r=2m$, as the Schwarzschild metric (2.5) has an artificial singularity at $r=2m$. The metric (2.5) also has a true singularity at $r=0$. The two types of singularities arise in (2.5) because one of the g^{ij} or g_{ij} becomes discontinuous (Foster & Nightingale, 1995). So, it represents regular patches at $0<r<2m$ or at $2m<r<\infty$. The region $0<r<2m$ is interior to the event horizon $r=2m$, a black hole region, and at $2m<r<\infty$ which is the region exterior to the event horizon. In the region $0<r<2m$ of the Schwarzschild metric (2.5); if $r\to0$, tidal forces become infinitely large, and the Riemann curvature scalar (the Kretschmann scalar), $K=R^{ijkl}R_{ijkl}=\frac{48m^2}{r^6}\to\infty$, where R_{ijkl} is the Riemann curvature tensor. This Kretschmann scalar shows that the point $r=0$ is an infinitely dense point-mass real space-time physical singularity which is produced due to irresistible gravitational collapse. But at the event horizon $r = 2m$ we find a coordinate singularity that can be detached by the appropriate choice of the coordinates.

2.1 Extension of the Coordinate Singularity

The Schwarzschild space-time metric (2.5) can be extended by an appropriate choice of coordinates at $r = 2m$ to become analytic at the coordinate singularity. It is a singularity that is formed for unsuitable selection of coordinates. The maximal analytic extension of the metric (2.5) with $2m < r < \infty$ is called Kruskal-Szekeres extension (Kruskal, 1960; Szekeres, 1960). For null geodesics we can write $ds = 0$, $d\theta = 0$, and $d\phi = 0$, then (2.5) becomes,

$$dt = \left(1 - \frac{2m}{r}\right)^{-1} dr \,. \tag{2.6}$$

Integrating (2.6) we get,

$$t = \pm\left[r + 2m\ln\left(\frac{r}{2m} - 1\right)\right] + \text{constant}$$

$$t = \pm r^{*} + \text{constant} \tag{2.7}$$

$$r^{*} = r + 2m\ln\left(\frac{r}{2m} - 1\right) \text{ and}$$

$$\frac{dr}{dr^{*}} = 1 - \frac{2m}{r}$$

with $-\infty < r^{*} < \infty$, where '+' sign indicates outside of the horizon, and '–' sign indicates inside the horizon. Here r^{*} is the "tortoise coordinate", as we move to the Schwarzschild radius $r = 2m$; r changes more and more slowly with r^{*}, as $\frac{dr}{dr^{*}} \to 0$ (Regge & Wheeler, 1957).

In equation (2.5), the advanced and retarded null coordinates have the direction of null geodesics, and these can be represented by;

$$u = t - r^{*}, \quad w = t + r^{*} \tag{2.8}$$

where $u \to \infty$ and $w \to -\infty$. In Schwarzschild space-time metric, the ingoing and outgoing light rays are indeed at 45^0. With a detail calculation and dropping the asterisk (*) we can write (2.5) as;

$$ds^2 = -\frac{2m}{r} e^{-r/2m} e^{(w-u)/4m} dudw + r^2 d\sigma^2. \quad (2.9)$$

As $r \to 2m$ relates to $u \to \infty$ or $w \to -\infty$, we can introduce new coordinates U and W respectively by; $U = -\ln(u/4m)$, and $W = \ln(w/4m)$, where the horizon is at $u = 4m$, i.e., at $U = 0$ and at $w = 4m$ i.e., at $W = 0$.

Hence (2.9) becomes;

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} \exp\left(e^{W} - e^{-U} + W - U\right) dUdW + r^2 d\sigma^2. \quad (2.10)$$

From (2.10) we observe that singularity is absent at $U = 0$ or $W = 0$ which agrees with $r = 2m$. Hence, the coordinate singularity $r = 2m$ is considered as being regular points in disguise in the space-time manifold. Again, we consider a transformation by using $T = \frac{W+U}{2}$ and $X = \frac{W-U}{2}$ then (2.10) becomes;

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} \left[\exp(X+T) - \exp(X-T) + 2X\right]\left(dT^2 - dX^2\right) + r^2 d\sigma^2 \quad (2.11)$$

which is Kruskal-Szekeres form of Schwarzschild metric. Finally, the transformation (t, r) to (T, X) becomes;

$$X^2 - T^2 = -UW = -\ln(u/4m)\ln(w/4m), \quad (2.12)$$

From (2.12), $r > 0$ gives $X^2 - T^2 > -\text{positive constant}$. The boundary of the Kruskal–Szekeres extension (2.11) is given by the physical singularity at $r = 0$, which provides the two sheets of the hyperbola $T^2 - X^2 = \text{positive constant}$, and now we see that $r = 2m$ is singularity free (Ashtekar, 2005).

2.2 Geometrical Properties of the Schwarzschild Manifold

The geometrical representation of the Schwarzschild metric is given by the Kruskal diagram. It is divided into four portions (Figure 1). The maximal domain covers two portions I und I' of the exterior portion $2m < r < \infty$. The portions I and I' are asymptotically flat and possess same characteristics. Actually, there is no communication, even by light signals, between the portions I and I'.

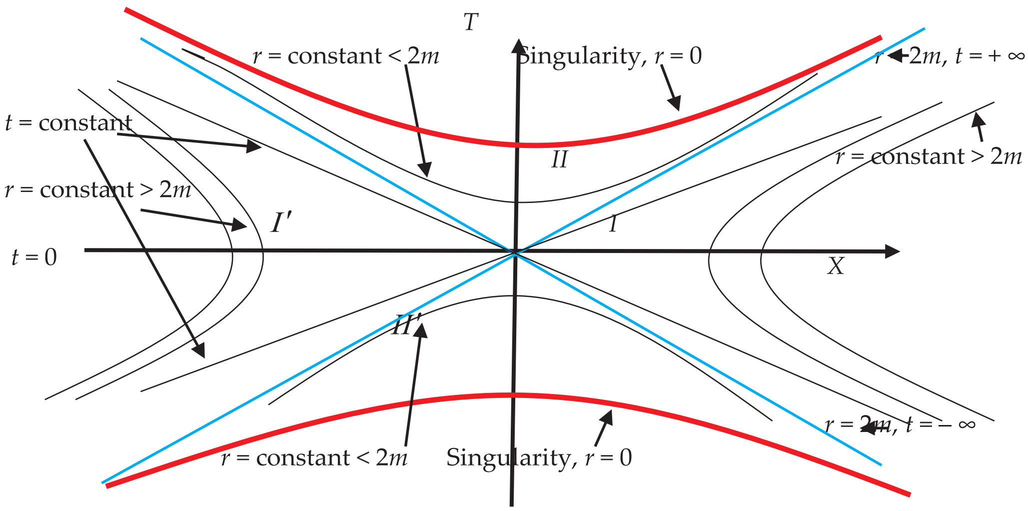

Figure 1: Kruskal diagram represents the Kruskal-Szekeres extension of the Schwarzschild metric. Each point on the figure indicates a 2-sphere.

The portion I' is another asymptotically flat universe on the other side of the Schwarzschild 'throat'. The interior portion II is called a black hole, and another interior portion II' is called a white hole. An observer who is now present in the portion II' of course originates from the singularity and must leave the portion II' again will reach to the portion *I* (Misner et al., 2005). Any outgoing observer from the portion I must travel to infinity. On the other hand, any ingoing observer must cross the null line $X = T$, and enters in the portion *II*. When a spectator falls into the portion *II*, a closed confined surface, cannot leave it, and within a finite proper time the spectator must face the singularity at $T^2 - X^2 =$ positive constant The explanation for region $r > 2m$ indicates to the portion *I*, which is considered as the outer gravitational field of a collapsing body where we always survive. Here the event horizon matches to the two lines $X = \pm T$ for $t \to \pm\infty$. Curves of $r = \text{constant}$ represent hyperbolas. In the region $r > 2m$, the hyperbolas fill the portions I and I' that are completely disconnected, and for $r > 2m$, they fill the other portions *II* and II'. The boundary of the portion II' is indicated by the past event horizon, on the other hand, the boundary of the portion *II* is indicated by the future event horizon.

The portion *I* is the Schwarzschild area which is disconnected by the horizon $r > 2m$, from the portions *II* and II'. The Eddington-Finkelstein coordinates (u, r), capture the portions *I* and *II*, and (v, r), capture the portions *I* and I' (Hawking & Ellis, 1973). Any observers in the portions *I* and I' can receive signals from the portion II' and send signals to the portion *II*.

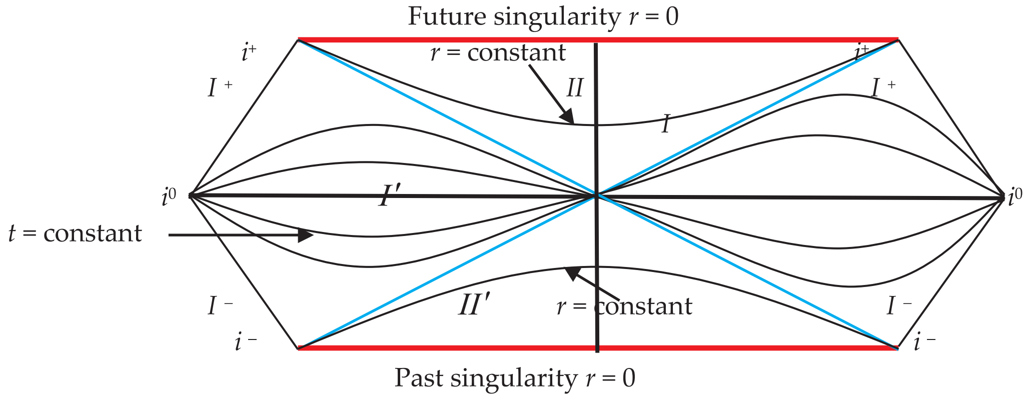

Figure 2: A conformal representation of the Schwarzschild geometry.

If an observer follows future-directed null rays he will reach in the portion *II*, for the following past-directed null rays he would reach the portion II'. Of course, portion II', a part of space-time, from which observers can escape to us, but we can never go back to there. The past singularity is indicated the start of the universe. If an observers had travelled spacelike geodesics, he would have been managed to the portion I'. The portion *II* is the black hole and if any observer travels from the portion *I* into *II*, it is not possible for him to return the portion *II*, and after a finite time must fall into the singularity at $r = 0$. We display a conformal illustration of Figure 1 as Figure 2.

An observer who is situated in the portion I', can send signals in the portions *II* and II', and finally falls in the singularity $r = 0$ at a finite proper time in the past. Similarly, a spectator who is stayed in the portion *II*, can receive signals from the portions *I* and I', and obviously will face the singularity at $r = 0$ in a proper finite time in future. In the portions *II* and I', world lines with r = constant are no longer remain timelike, but will be spacelike for $r < 2m$.

3. Friedmann, Robertson-Walker (FRW) Singularity

The FRW models are formed on the basis of the homogeneity and isotropy of the universe. The non-static spherically symmetric line element in the moving coordinate system is presented by,

$$ds^2 = dt^2 - e^{\mu(r,t)}\left(dr^2 + r^2 d\sigma^2\right) \tag{3.1}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

If we write,

$$dl^2(t_1) = e^{\mu(r,t_1)}\left(dr^2 + r^2 d\sigma^2\right)$$
$$dl^2(t_2) = e^{\mu(r,t_2)}\left(dr^2 + r^2 d\sigma^2\right). \tag{3.2}$$

The ratio of $dl(t_1)$ and $dl(t_2)$ is independent of space coordinates if and only if $\mu(r,t)$ has the form,

$$\mu(r,t) = f(r) + g(t). \tag{3.3}$$

Then (3.2) becomes,

$$ds^2 = dt^2 - e^{f(r)+g(t)}\left(dr^2 + r^2 d\sigma^2\right). \tag{3.4}$$

Now, we write, $\dot{\mu} = \frac{\partial \mu}{\partial t}$, $\mu' = \frac{\partial \mu}{\partial r}$, etc., by calculating surviving Γ's we write the Ricci tensor R_{ij} as;

$$R_{11} = f'' + \frac{f'}{r} - e^{\mu}\left(\frac{1}{2}\ddot{g} + \frac{3}{4}\dot{g}^2\right)$$

$$R_{22} = r^2\left[\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2r}f' - e^{\mu}\left(\frac{1}{2}\ddot{g} + \frac{3}{4}\dot{g}^2\right)\right] \tag{3.5}$$

$$R_{33} = \sin^2\theta R_{22}$$

$$R_{44} = \frac{3}{2}\ddot{g} + \frac{3}{4}\dot{g}^2$$

$$R_{ij} = 0 \text{ if } i \neq j.$$

By using $R^i_j = g^{ik}R_{jk}$ we get,

$$R^1_1 = \frac{1}{2}\ddot{g} + \frac{3}{4}\dot{g}^2 - e^{-\mu}\left(f'' + \frac{f'}{r}\right)$$

$$R^2_2 = R^3_3 = r^2\left[\frac{1}{2}\ddot{g} + \frac{3}{4}\dot{g}^2 - e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{4}f'^2 + \frac{3}{2r}f'\right)\right] \tag{3.6}$$

$$R^4_4 = \frac{3}{2}\ddot{g} + \frac{3}{4}\dot{g}^2.$$

The Ricci scalar,

$$R = R_i^i = R_1^1 + R_2^2 + R_3^3 + R_4^4$$

$$= 3\left(\ddot{g} + \dot{g}^2\right) - 2e^{-\mu}\left(f'' + \frac{1}{4}f'^2 + \frac{2}{r}f'\right).$$

Substituting these values in the Einstein's field equation (1.1) we get,

$$8\pi T_1^1 = \frac{1}{2}R - R_1^1 - \Lambda$$

$$= \ddot{g} + \frac{3}{4}\dot{g}^2 - e^{-\mu}\left(\frac{1}{4}f'^2 + \frac{2}{r}f'\right) - \Lambda \cdot \qquad (3.7a)$$

$$8\pi T_2^2 = 8\pi T_3^3 = \frac{1}{2}R - R_2^2 - \Lambda$$

$$= \ddot{g} + \frac{3}{4}\dot{g}^2 - e^{-\mu}\left(\frac{1}{2}f'' + \frac{f'}{2r}\right) - \Lambda . \qquad (3.7b)$$

$$8\pi T_4^4 = \frac{1}{2}R - R_4^4 - \Lambda$$

$$= \frac{3}{4}\dot{g}^2 - e^{-\mu}\left(f'' + \frac{1}{4}f'^2 + \frac{2}{r}f'\right) - \Lambda . \qquad (3.7c)$$

$$8\pi T_j^i = 0 \text{ if } i \neq j .$$

Further, the assumption of spherical isotropy of 3-spaces indicates that the longitudinal and transverse stresses must be equal, i.e., $T_1^1 = T_2^2 = T_3^3$. Hence, (3.7) indicates,

$$f'' - \frac{1}{2}f'^2 - \frac{f'}{r} = 0 . \qquad (3.8)$$

Integrating (3.8) we get,

$$\frac{df}{dr} = k_1 r e^{f/2} \qquad (3.9)$$

where k_1 is integration constant. Again integrating (3.9) we get,

$$e^f = \frac{\frac{1}{k_2^2}}{\left(1+\frac{r^2}{4S_0^2}\right)^2}. \tag{3.10}$$

This may be written as,

$$e^{f(r)} = \left(1+\frac{kr^2}{4S_0^2}\right)^{-2} \tag{3.11}$$

where $k = 1, 0, -1$.
Using (3.11), equation (3.4) can be written as,

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1+\frac{kr^2}{4S_0^2}\right)^2}\left(dr^2 + r^2 d\sigma^2\right). \tag{3.12}$$

This line element is known as RW line element, which can be expressed as,

$$ds^2 = dt^2 - \frac{S_0^2 e^{g(t)}}{\left(1+\frac{kr^2}{4}\right)^2}\left(dr^2 + r^2 d\sigma^2\right). \tag{3.13}$$

In (t, r, θ, ϕ) coordinates the FRW line element (3.13) can be represented as (Mohajan, 2013c);

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\sigma^2\right] \tag{3.14}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. Here $S(t)$ represents cosmic scale factor, and $k = +1, 0, -1$. For $k = 0$, the 3-space becomes flat and (3.14) indicates Einstein-de Sitter static universe, when $k = +1$ and $k = -1$ the 3-space represents positive and negative constant curvatures; that conglomerate the closed and open Friedmann models, respectively (Figure 3). Form (3.14) we get;

$$\dot{\rho} + 3(\rho + p)\frac{\dot{S}}{S} = 0. \tag{3.15}$$

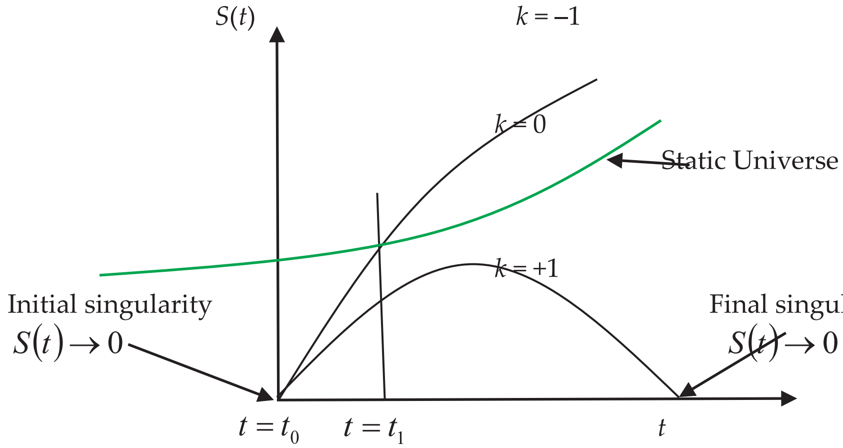

Figure 3: Nature of the FRW model. In static universe, $S(t) \neq 0$ at $t = 0$.

The energy-momentum tensor T^{ij} can be written as;

$$T^{ij} = \varepsilon_0 u^i u^j \tag{3.16}$$

where ε_0 is the proper density of matter in the absence of pressure. A perfect fluid is categorized by pressure $P = P(x^i)$, then energy-momentum tensor can be written as (Carroll, 2004);

$$T^{ij} = (\varepsilon + P)u^i u^j - Pg^{ij}. \tag{3.17}$$

We consider the matter of the universe as an ideal fluid, then by (3.16) and (3.17), also solving (3.14), we have;

$$\frac{3\ddot{S}}{S} + 4\pi(\varepsilon + 3P) = 0, \text{ and} \tag{3.18}$$

$$\frac{3\dot{S}^2}{S^2} - \left(8\pi\rho - \frac{3k}{S^2}\right) = 0 \tag{3.19}$$

where for convenient we have considered $\Lambda = 0$.

We have three variables S, ε, P but only two equations (3.18) and (3.19). So, one additional equation is necessary to obtain the solution that can be presented by the equation of state, $P = P(\varepsilon)$. If $\varepsilon > 0$ and $P \geq 0$, then (3.18) indicates, $\ddot{S} < 0$. So, (3.19) provides, $\dot{S} =$ constant, and $\dot{S} > 0$ specifies expanding universe, and $\dot{S} < 0$ designates contracting universe.

American astronomer Edwin Hubble (1889-1953) observed the red-shifts of the galaxies. He observed that galaxies are moving away from us with a velocity proportional to their distances from us. From this observation he revealed that the universe is expanding, which verifies the prediction of the general theory of relativity. As the universe is expanding, so, $\dot{S} > 0$; by (3.18) and (3.19) we get, $\ddot{S} < 0$. As a result, $\dot{S}$ is a decreasing function, which indicates at earlier times the universe must have expansion at a faster rate as compared to the present rate of expansion. But, if the expansion would be constant rate at all times, i.e., at past, at present, and in future, then,

$$\left(\frac{\dot{S}}{S}\right)_{t=t_0} \equiv H_0 . \tag{3.20}$$

Here H_0 is considered as the Hubble constant. We have $\frac{\ddot{S}}{S} < 0$, $S(t_0) > 0$ and $\frac{\dot{S}(t_0)}{S(t_0)} > 0$, here $t = t_0$ indicates present time. Hence, scale factor $S(t)$ must be concave downwards and ultimately $S(t)$ touch the t-axis at a finite time $t = t_1$ in future (Figure 4). This time is considered the big crunch, which is the end point of the universe.

Hence, at $t = 0$, we have;

$$S(0) = 0 . \tag{3.21}$$

The event $t = 0$ is the beginning of the universe. Also, for $t = t_1$, i.e., at a finite time in the future the universe will again face $S(0) = 0$. Both at $t = 0$ and $t = t_1$ the nature of the universe become $S = 0$. The past singularity at $t = 0$ is called the "big bang". Similarly, the future singularity at $t = t_1$ is called the "big crunch".

4. Singularities Due to Gravitational Focusing

In all the stars, hydrogen is used as fuel, and after burn creates helium. In this process the volume of the star contracts due to inward gravitational force. After a certain period, contraction process will halt and it will achieve a long stationary state due to the thermal and radiation pressures. The length of the static state may be even billions of years, according to the initial mass of the star. The mass of the sun is denoted by $M_O \approx 2\times10^{33}$ gm, if M_S is the mass of the star, then if $M_S < M_O$, this period becomes more than 10^{10} years. On the other hand, if $M_S > 10M_O$ the burning period will be less than $2\text{x}10^7$ years. This scenario indicates that the weightier stars burn out their nuclear fuel very quickly than the lighters. After converting all hydrogen to helium during the burning process collapse procedure cannot stop if the star is adequately massive. Then thermonuclear reactions start and helium converts into carbon. Indian scientist Subrahmanyan Chandrasekhar (1910-1995) introduced greatest mass for a non-rotating star to attain a white dwarf stationary state as (Chandrasekhar, 1983);

$$M_c = 1.4\left(\frac{2}{\mu_c}\right)^2 M_O \tag{4.1}$$

where μ_c is the constant mean molecular weight per electron. The extremum mass for non-rotating white dwarf may be in $1.0M_O - 1.5M_O$, subject to the conformation of matter, and for neutron stars the corresponding figure increase to $1.3M_O - 2.7M_O$ (Arnett & Bowers, 1977).

5. Black Hole Formation

A sufficiently massive star continues its contraction for gravitational force until it reaches the space-time singularity. At this stage no equilibrium state is possible and eventually creates a black hole, which covers the space-time singularity. In the Schwarzschild metric, the region $r < 2m$ is considered as trapped surface and no signal or particle come outside. Consequently, a black hole is formed in that region. The hypersurface $r = 2m$ is considered as the "*absolute event horizon*" of the space-time [15]. At the singularity $r = 0$, the curvature and density become infinite and singularity becomes invisible to the external observers (Joshi, 1996).

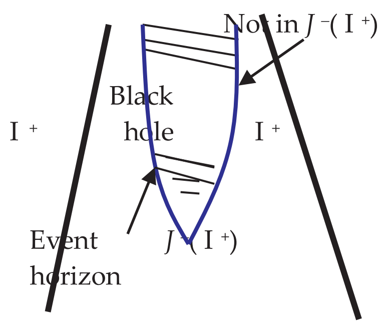

Figure 4: The black hole structure.

After crossing region $r = 2m$, the individual experiences as a 'tidal force' and realizes that he is forced to enter regions of much smaller r (Penrose, 1970). In an asymptotic flat space-time *M*, a black hole region is defined by (Figure 4);

$$B = M - J^{-}(\mathrm{I}^{+}) \tag{5.1}$$

which is closed in *M*, and the event horizon is confined in *B*. The boundary of *B* in *M*, which is also considered as the event horizon as;

$$H = \dot{J}^{-}(\mathrm{I}^{+}) \cap M. \tag{5.2}$$

6. Conclusions

In this study we have analyzed that the Schwarzschild solution plays an important role for identifying and explaining event horizons, and space-time singularities. There is a coordinate singularity at $r = 2m$ which can be removed by Kruskal-Szekeres maximal extension. But $r = 0$ is a real curvature singularity that cannot be removed as like that of $r = 2m$. In this study we have discussed the Kruskal-Szekeres maximal extension to remove coordinate singularity. On the other hand, in FRW model there was a real curvature singularity in the past where $t = 0$ and this is called big bang singularity. By the mathematical procedure and physical interpretation, we observe that there may arise a future curvature singularity in FRW model which we call big crunch.

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