Ramifications of Laplace's Equation and Electrostatic Fields

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(A.) Abstract: *The basic field solutions for electrostatic field in free space can be expressed in terms of a scalar potential function. The electrostatic field is conservative everywhere. The total scalar potential at any arbitrary point, on the other hand, is simply the scalar sum of the contributions produced by each of the independent elemental source points. The differential elements of source charge may consist of several types of contributions that range from volume, surface, and line charge densities to point charges. Thus, a concrete expression can be constructed for electrostatic field when the distribution of charges is known throughout the space. A much more typical case arises when the static charge distribution is known over some limited region of space only. Differential field laws can be applied at all points within the region and thereby the conditions for the mathematical acceptability of the static field can be established. Poisson's equation expresses the condition of mathematical acceptability for electrostatic potential in the regions where volume charge density is not zero. Laplace's equation, on the other hand, expresses the condition of mathematical acceptability for electrostatic potential in all charge free regions. An electrostatic problem involving linear, isotropic, and homogeneous dielectric reduces, therefore, to finding solutions of Laplace's equation in each medium, and joining the solutions in various media by means of boundary conditions. Hence, it is needed for one to understand some mathematical charateristics of Laplace's equation in order to know exactly what information is necessary for a solution of Poisson's or Laplace's equation.*

(B.) Introduction: The elegance and beauty of Maxwell's formulation of the law of electromagnetism for source charge can quickly disappear in a cloud of involved mathematical technique as soon as one attempts to determine the fields produced by specific systems. It is an aim here, to highlight the most common of the basic field solutions for the static field in the free space. Since the static field **E** is constrained to be conservative, we know that we can always express **E** in terms of a scalar potential function [Ref-5: p-47] Φ:

$$\mathbf{E} = -\nabla\Phi \quad \ldots (1)$$

The total scalar potential at any arbitrary point P is simply the scalar sum of the contributions $d\Phi_p$ produced by each of the elemental source points dq, that is,

$$\Phi_p = (1/4\pi\varepsilon_0)\int_{\text{all charges}}(dq_Q/r_{QP}) \quad \ldots \quad (2)$$

The differential elements of source charge dq may consist of several types of contributions that range from volume, surface, and line charge densities to point charges. Thus, for an arbitrary distribution of charge, the total electrostatic potential at point P is then simply the linear superposition of each of the separate charge contributions, as given by-

$$\Phi = (1/4\pi\varepsilon_0)\ [\int_V(\rho_Q dv_Q/r_{QP}) + \int_S(\eta_Q da_Q/r_{QP}) + \int_L(\lambda_Q ds_Q/r_{QP}) + \sum_K q_k/r_{QP} \quad \ldots(3)$$

where, for simplicity of notation, the subscript P has been dropped from the potential at point P. It is to be noted that Φ is a function of the coordinates of the point P alone. On the other hand, ρ_Q, η_Q, λ_Q, and q_k depend solely on the coordinates of the point Q, while-

$$r_{QP} = [(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]^{(1/2)} \quad \ldots(4)$$

is a symmetrical function of the coordinates of P and Q. We observe, in this regard, that the differentiation involved in computing the potential gradient must be carried out with respect to the coordinates of the point P, with the coordinates of Q constant. Thus,

$$\mathbf{E} = -\nabla_P = -\nabla_P[(1/4\pi\varepsilon_0)\int(dq_Q/r_{QP})] = -(1/4\pi\varepsilon_0)\int[\nabla_P(1/r_{QP})]dq_Q \ldots(5)$$

And in the Cartesian form this becomes:

$$\mathbf{E}=(1/4\pi\varepsilon_0)[\int_V(\rho_Q dv_Q/r^2_{QP})\mathbf{i}_{QP}+\int_S(\eta_Q da_Q/r^2_{QP})\mathbf{i}_{QP}+\int_L(\lambda_Q ds_Q/r^2_{QP})\mathbf{i}_{QP}+\sum_K(q_k/r^2_{QP})\mathbf{i}_{QP} \quad \ldots(6)$$

The above equation is the expression for electrostatic field when the distribution of charges is known throughout the space.

A much more general case that arises when the static charge distribution is known (over some limited region of space only) will now be considered. We can apply differential field laws at all points within the region and thereby can establish the conditions for the mathematical acceptability of **E**. We also know that the static **E** must be conservative everywhere [Ref-5: p-47],

$$\nabla \times \mathbf{E} = 0 \ldots(7)$$

and that it's divergence in the presence of volume charge density ρ, is given [Ref-5: p-46] at each point by-

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \ \ldots \quad (8)$$

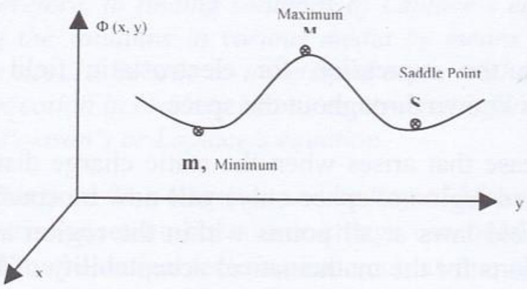
where ε_0 is the permittivity of vacuum. Thus equation (1) and (8) together gives:

$$\nabla \cdot \mathbf{E} = -\nabla \cdot (\nabla\Phi) = -\nabla^2\Phi = \rho/\varepsilon_0 \Leftrightarrow \nabla^2\Phi = -\rho/\varepsilon_0 \quad \ldots \quad (9)$$

where $\nabla^2 = \nabla \cdot \nabla$ is known as the Laplacian operator and the above equation is known as Poisson's equation. If $\rho = 0$, eqn. (9) reduces to $\nabla^2\Phi = 0$. This relation known as Laplace's equation, expresses the condition of mathematical acceptability for electrostatic potential in all charge-free regions of space. An electrostatic problem involving linear, isotropic, and homogeneous dielectrics reduces, therefore, to finding solutions of Laplace's equation in each medium, and joining the solutions in various media by means of boundary conditions. Very often, however, exact solutions of Laplace'e equation either can not be found or else take on forms too unmanageable for simple analysis. It is necessary for one to understand some (mathematical) attributes of Laplace's equation in order to know exactly what information is necessary for a solution of Poisson's or Laplace's equation.

(C.) A Maximum-Minimum Theorem: *A function Φ that satisfies Laplace's equation within a region V can attain neither a maximum nor a minmum value within V, and thus Φ must assure its largest and smallest values in V on the surface S enclosing V.*

This theorem [Ref-6: p-666, 726-744] can be proven by noticing that the second partial derivatives of Φ with respect to x, y, and z must all be positive at a point where Φ has a minimum, and must all be negative at a point where Φ has a maximum, as illustrated here for a two-dimensional potential. This, however, is impossible, as because their sum must



be equal to zero by Laplace's equation. This theorem permits us to conclude that the static **E** field must be equal to zero at all points within a charge-free region completely enclosed by a current-free conductor. This follows from the fact that the largest and the smallest values assumed by potential in such a region must be equal, since the potential is forced to be constant on the surface of an isolated conductor. Thus, with neither a maximum nor a minimum allowed within the given volume and no variation at all on the surrounding surface, the potential is constrained to be constant throughout the entire region and **E**= -**∇**Φ = **0**.

(D.) The Uniqueness Theorem: A second conclusion can be drawn from the lack of a maximum or minimum value of a solution of Laplace's equation within a charge-free region. Specifically, *the electric field in any charge-free region is uniquely specified by the values assumed by the potential on the surface enclosing the region.*

This follows from the fact that if there were two potential functions (Φ_1 and Φ_2) that satisfied Laplace's equation within the region and that assumed the same values on its surface, their difference would also satisfy Laplace's equation and would vanish on the surface. This suggests that the potential within a charge-free region should be uniquely specified by the values assumed on the surface S of the region by either the potential Φ or by the component of the current density to normal to S, that is $J_n = \mathbf{n}.\mathbf{J}$, where $\mathbf{n}$ is the unit vector normal to S. On the other hand, $\mathbf{J} = \sigma\mathbf{E} = -\sigma\nabla\Phi$ within a conductor (of conductivity σ), so that J_n is directly proportional to the derivative of Φ normal to S. That is, $J_n = -\sigma(\mathbf{n}.\nabla\Phi) = -\sigma(\partial\Phi/\partial n)$.

Thus, we must expect the potential within a charge-free region is also to be uniquely specified by the values given for either Φ or $\partial\Phi/\partial n$ at each point on the surface S that encloses the region. This is, indeed, the case. In fact, stated more precisely, this conclusion forms the uniqueness theorem that is included below, namely: '*Any solution Φ of Laplace's equation in a volume V is uniquely specified when either the value of Φ or $\partial\Phi/\partial n$ is given at all points on the surface S enclosing V*'.

To derive this uniqueness theorem, however, we need a relation known as Green's theorem [Ref-2: p-628, 683; Ref-5: p-19; Ref-6: p-834]. It states that, "*Given an arbitrary scalar function Φ which together with its gradient and Laplacian is non-singular within a volume V and on the surface S enclosing V, then-*

$$\int_s \Phi\nabla\Phi.\mathbf{da} = \int_v (\|\nabla\Phi\|^2 + \Phi\nabla^2\Phi)dv \text{''} \quad \ldots \quad (10)$$

This theorem follows directly from the divergence of the product of Φ and $\nabla\Phi$,

$$\nabla.(\Phi\nabla\Phi) = (\nabla\Phi.\nabla\Phi) + \Phi\nabla^2\Phi \quad \ldots \quad (11)$$

If we integrate both sides of eqn. (11) over the volume V and then apply Gauss's theorem, we arrive at the desired relation, that is,

$$\int_v \nabla.(\Phi\nabla\Phi)dv = \int_s (\Phi\nabla\Phi).\mathbf{da} = \int_v (\|\nabla\Phi\|^2 + \Phi\nabla^2\Phi)dv \quad \ldots(12)$$

We now observe that if the function Φ in equation (12) represents a solution of Laplace's equation, $\nabla^2\Phi = 0$ at all points within V, eqn. (12) reduces to –

$$\int_s \Phi(\partial\Phi/\partial n)da = \int_v \|\nabla\Phi\|^2 dv \ldots \quad (13)$$

since, $(\nabla\Phi).\mathbf{da} = (\nabla\Phi).\mathbf{n}da = (\partial\Phi/\partial n)da$ where $\mathbf{n}$ is the outward unit vector normal to S. The left hand side of eqn. (13) vanishes whenever Φ or $\partial\Phi/\partial n$ is zero at all points on S. When that happens the integrand $\|\nabla\Phi\|^2$ on the right side of eqn. (13) must then vanishes at all points of V. Then-

$$\nabla\Phi = 0 \quad \ldots(14)$$

at all points in V with the result that Φ must be constant throughout V. We can conclude, therefore, that: '*Any solution Φ of Laplace's equation must be constant throughout a region V whenever either Φ or its normal derivative ∂Φ/∂n vanishes at all points on the surface S enclosing V*'.

Finally, to prove the uniqueness theorem, we need only to note that if there were two different solutions Φ_1 and Φ_2 of Laplace's eqn. in V that possessed the same values of either Φ or ∂Φ/∂n at each point on the surface S enclosing V, then their difference,

$$\Phi = \Phi_1 - \Phi_2 \quad \ldots. \quad (15)$$

would also be a solution of Laplace's equation that would have either Φ or ∂Φ/∂n equal to zero at all points on S. It follows from eqn. (13) and (14) that Φ must be a constant throughout the volume V and, therefore, Φ_1 and Φ_2 can differ at most by a constant C,

$$\Phi_1 = \Phi_2 + C \quad \ldots \quad (16)$$

We can conclude then that a solution of Laplace's eqn. is uniquely specified within a region V when either its value or value of (normal derivative) ∂Φ/∂n is given at all points on the surface S enclosing V.

It follows from eqn. (16) that the two electric fields derivable from Φ_1 and Φ_2 must be identical at all points in V,

$$\mathbf{E}_1 = -\nabla\Phi_1 = -\nabla\Phi_2 = \mathbf{E}_2 \quad \ldots(17)$$

Thus, the uniqueness theorem also states that the electric field **E** derived from a solution of Laplace's eqn. within a region V is, uniquely specified when either the value of (potential) Φ or its normal derivative ∂Φ/∂n is given at all points on the surface S enclosing V.

Thus, in addition to necessary information for solution of Laplace's equation we also know when to stop working. Once we find a solution that meets all of the necessary boundary conditions required by the uniqueness theorem S, we are assured of having the only (that is, the unique) solution. We need look no further.

(E.) Solutions of Laplace's Equation in Rectangular Coordinates: If the potential is a function of (3-D) rectangular coordinates, Laplace's equation is given [Ref-5: p-17] by-

$$\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 = 0 \quad \ldots(18)$$

Linear partial differential equations of this type are solvable by expressing (electrostatic potential) Φ as the product of the functions, each involving only one of the three variables. In particular, on substituting-

$$\Phi(x,y,z) = X(x)Y(y)Z(z)\ldots(19)$$

where X is a function of x alone, Y is a function of y alone, and Z is a function of z alone, into Laplace's equation and dividing by Φ, we obtain-

$$(1/X)d^2X/dx^2 + (1/Y)d^2Y/dy^2 + (1/Z)d^2Z/dz^2 = 0 \quad \ldots \quad (20)$$

It is clear from the above equation that if the sum of these terms is to be equal to zero for all values of x,y,zϵ**R**, each of three terms must either be equal to zero or, at best, a constant.

Case I. Trivial Solutions: The simplest or "trivial" class of solutions of Laplace's equations in Cartesian coordinates can be found by setting each term in eqn. (20) separately equal to zero,

$$d^2X/dx^2 = 0;\ d^2Y/dy^2 = 0;\ d^2Z/dz^2 = 0 \quad \ldots \quad (21)$$

Each of three resulting second-order linear differential equations can readily be integrated to yield the solutions:

$$X = A + Bx,\ Y = C + Dy,\ \ Z = E + Fz \quad \ldots(22)$$

Where A, B, C, D, E, and F are arbitrary constants. We then arrive at the family of trivial solutions of Laplace's equation in Cartesian coordinates by substituting eqn. (22) into eqn. (19):

$$\Phi(x,y,z) = (A + Bx)(C + Dy)(E + Fz)\ldots \quad (23)$$

This solution may be applied to the case where three conducting planes (xy, yz, zx) intersect at right angles. If the three planes are all at the same electrostatic potential then-

$$\Phi(x,y,z) = ACE + BDFxyz \quad \ldots \quad (23A)$$

Case II. General Solutions: The general class of solutions of Laplace's equation in Cartesian coordinates follows from the setting of each of three terms in eqn. (20) separately equal to a constant, subject only to the restriction that the sum of the three constants must be equal to zero. Thus, we have-

$(1/X)d^2X/dx^2$ $\pm K_x^2$	=	$(1/Y)d^2Y/dy^2$ $\pm K_y^2$	=	$(1/Z)d^2Z/dz^2$ $\pm K_z^2$	=	… (24A,B,C)

where K_x, K_y, $K_z\epsilon$**R** are arbitrary constants such that:

$$\pm K_x^2 \pm K_y^2 \pm K_z^2 = 0 \ldots(25)$$

Each of the three second-order linear differential equations given by eqn. (24) has well-known simple solutions. For X(x), for instance, we have-

$X(x)=A[\exp(ik_xx)]+B[\exp(-k_xx)]$ or $A'[\mathrm{Sin}(k_xx)]+B'[\mathrm{Cos}(k_xx)]$; for $-k_x^2$ ...(26)

for negative sign in eqn. (24A), and-

$X(x)=C[\exp(k_xx)]+D[\exp(-k_xx)]$ or $C'[\mathrm{Sinh}(k_xx)]+D'[\mathrm{Cosh}(k_xx)]$; for $+k_x^2$ ...(27)

for the positive sign. Similar results apply to the other two functions Y and Z with (k_yy) and (k_zz), respectively, replacing (k_xx) in eqns. (26) and (27).

Nota Bene: $\exp(\pm ik_xx) = \mathrm{Cos}(k_xx) \pm i\mathrm{Sin}(k_xx)$ and $\exp(\pm k_xx) = \mathrm{Cosh}(k_xx) \pm \mathrm{Sinh}(k_xx)$

The set of general solutions of Laplace's equation in Cartesian coordinates is then given by the product of X, Y, and Z as in eqn. (19). If no one of the three constants k_x, k_y, and k_z is zero, then appropriate solutions are:

$$\Phi(x,y,z) = \{\exp(\pm jk_xx)\}\{\exp(\pm jk_yy)\}\{\exp(\pm k_zz)\} \quad \ldots \quad (28)$$

in which the brace notation is intended to signify: $\exp(\pm k_xx) = A[\exp(k_zz)]+B[\exp(-k_zz)]$ with A and B constatnts; etc.

All of these solutions may be listed in compact form as the product of three sets of five functions, one set for each of the three rectangular coordinates, as:

$$\Phi = X(x)Y(y)Z(z)=\{\mathrm{Sin},\ \mathrm{Cos},\ e^{\pm},\ \mathrm{Sinh},\ \mathrm{Cosh}\}k_xx\{\mathrm{Sin},\ \mathrm{Cos},\ e^{\pm},\ \mathrm{Sinh},\ \mathrm{Cosh}\}k_yy\{\mathrm{Sin},\ \mathrm{Cos},\ e^{\pm},\ \mathrm{Sinh},\ \mathrm{Cosh}\}k_zz \quad \ldots \quad (29)$$

where A, k_x, k_y, and k_z are constants. Here terms of the form $e^{\pm j}$ have been omitted, since the physically realizable static fields must be real. Each of the general solutions of the Laplace's equation in Cartesian coordinates can now be formed by simply taking the product of the constants A and one term from each of the three brackets in eqn. (29), with k_xx, k_yy, and k_zz as the respective arguments, such that:

$$\pm k_x^2 \pm k_y^2 \pm k_z^2 = 0 \quad \ldots \quad (30)$$

where the negative signs apply to the first two terms in each bracket, and positive signs apply to the last three terms in each bracket.

To prove that the functions so taken from eqn. (29) are, in fact, solutions of Laplace's equation, wee need only to show that they satisfy Lalace's equation, we find that the second derivative of each of five terms in each bracket yields either $k_j^2\Phi$ or $-k_j^2\Phi$ (where j = x, y, or z), depending on whether the specific term chosen comes from the first two or the last three terms in each bracket, respectively. Thus, we get-

$$\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 = (\pm k_x{}^2 \pm k_y{}^2 \pm k_z{}^2)\Phi = 0 \quad \ldots(31)$$

It then follows that eqn. (30) is both a necessary and a sufficient condition for any of the non-zero functions included in eqn. (29) to be a solution of Laplace's equation.

(F.) Solutions of Laplace's Equation in Cylindrical Coordinates: Laplace's equation in the cylindrical coordinates r[1], θ, z is [Re-5: p-52, 59]-

$$\nabla^2\Phi = (1/r)(\partial/\partial r)[r(\partial\Phi/\partial r)] + (1/r^2)(\partial^2\Phi/\partial\theta^2) + \partial^2\Phi/\partial z^2 = 0 \quad \ldots(32)$$

Separating by means of product functions,

$$\Phi(r, \theta, z) = [R(r)][\Theta(\theta)][Z(z)], \ldots(33),$$

we obtain,

$r(d/dr)[r(dR/dr)]+(K^2r^2-n^2)R = 0$	$d^2\Theta/d\theta^2+n^2\Theta = 0$	$d^2Z/dz^2-K^2Z = 0$	(34A,B,C)

where K and n are the separation parameters. Eqn. (34A) is known as Bessel's equation, and its solutions are called Bessel functions. The character of the solution will depend markedly on the sign of separation constant, i.e., on whether n and K are real or imaginary. If solutions are desired which are single valued in azimuthal angle θ, then the solutions must be periodic in θ, and n must be an integer. To solve, eqn. (34A) can be written as:

$$r^2(d^2R/dr^2)+r(dR/dr)+(K^2r^2-n^2)R=0, \Leftrightarrow R''+(1/r)R''+[K^2 - (n^2/r^2)]R = 0 \quad \ldots(35)$$

However, no integer restriction exists on the allowable values of K, indeed, since K^2 can be any real constant, K can be any pure real number or any pure imaginary number. If both K and n are zero, eqn. (35) has simple solution: $R(r) = a \ln(r) + b$, $n = k = 0$ whereas if only k equals zero, $R(r) = \{r^{\pm n}\}$; $k = 0$.

For $k \neq 0$, it is best to proceed by introducing the substitution variable (such as) $s = kr$, which converts eqn. (35) to-

$$(d^2R/ds^2) + (1/s)(dR/ds) + [1-(n^2/s^2)]R = 0 \quad \ldots (36)$$

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[1] The cylindrical and rectangular coordinates are related in the way: x = r cos(θ),. y = r sin(θ).

It is routine to check that this differential equation has a regular singular point at zero, suggesting that we try Frobenius solution:

$$R = \sum_{j=0}^{\infty} a_j s^{j+\lambda} \ldots \quad (37)$$

Substituting (37) into (36) we get:

$a_0[\lambda(\lambda-1)+\lambda-n^2]s^{\lambda}+a_1[(\lambda+1)\lambda+(\lambda+1)-n^2]s^{\lambda+1}+\sum_{j=2}^{\infty}\{[(j+\lambda)(j+\lambda-1)+(j+\lambda)-n^2]a_j+a_{j-2}\}s^{j+\lambda}$.

In this, the co-efficient of each power of s must be zero. From the co-efficient of s^{λ} we get the indicial equation: $\lambda^2 - n^2 = 0$, assuming $a_0 \neq 0$. This has roots $\lambda = n, -n$ and leads to two cases.

Case I, n=0: Now $\lambda=0$ is a repeated root of the indicial equation. To get a solution, set $\lambda=n=0$ and obtain the recurrence relation: $j^2a_j+a_{j-2} = 0$; $j = 2, 3, 4, \ldots$ from the co-efficient of $s^{j+\lambda}$ in the above summation. Then, $a_j = -a_{j-2}/j^2$ for $j= 2, 3, 4, \ldots$. From this we easily find the even indexed terms: $a_{2j} = [(-1)^j a_0]/[2^2 4^2 \ldots (2j)^2]$ for $j= 2, 3, 4, \ldots$ The recurrence relation gives no information about a_1. However, upon setting $\lambda=n=0$ in the co-efficient of $s^{\lambda+1}$ above, we get the co-efficient of $s^{\lambda+1}$ to be just a_1, and hence conclude that $a_1=0$. From recurrence relation, then: $a_1= a_3 =a_5 = a_5= \ldots =0$ and all odd-indexed co-efficients are, therefore, zero. Thus, one solution is,

$R_1= \sum_{j=0}^{\infty} a_{2j}s^{2j} = a_0\sum_{j=0}^{\infty}\{[(-1)^j]/[2^2 4^2\ldots(2j)^2]\}s^{2j} = a_0\sum_{j=0}^{\infty}\{[(-1)^j]/[2^{2j}(j!)^2]\}s^{2j} = a_0\sum_{j=0}^{\infty}\{[(-1)^j]/[(j!)^2]\}(s/2)^{2j}$.

When we choose $a_0=1$, this solution is called the Bessel function of the first kind of order zero, and this denoted $J_0(s)$. Thus,

$J_0(s) = \sum_{j=0}^{\infty}\{[(-1)^j]/[(j!)^2]\}(s/2)^{2j} = 1 - (s^2/2^2) + (s^4/2^2 4^2) - (s^6/2^2 4^2 6^2) + (s^8/2^2 4^2 6^2 8^2) - \ldots$ is a first solution of eqn. (36). A graph of $R(r) = J_0(s)$ is shown:

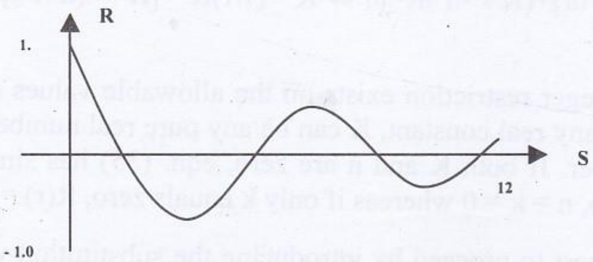


Table 1 (next), and table 2 (next) gives some values of some positive roots of the equation $J_0(s) = 0$ and $J_0(s)$ for $0 \leq s \leq 10$ respectively.

Table 1: The First 30 Positive Roots of $J_0(s) = 0$ (in Increasing Order from Zero)				
Root	**Root**	**Root**	**Root**	**Root**

#	Value	#	Value	#	Value	#	Value	#	Value
1	2.40483	7	21.21164	13	40.05843	19	58.90698	25	77.75602
2	5.52008	8	24.35247	14	43.19979	20	62.04846	26	80.89756
3	8.63573	9	27.49348	15	46.34119	21	65.18996	27	84.03909
4	11.79153	10	30.63461	16	49.48261	22	68.33146	28	87.18063
5	14.93092	11	33.77582	17	52.62405	23	71.47298	29	90.32217
6	18.07106	12	36.91709	18	55.76551	24	74.61450	30	93.46371

Table 2: Some Values of $R = J_0(s)$ for $0 \leq s \leq 10$

#	S	$J_0(S)$	#	S	$J_0(S)$	#	S	$J_0(S)$	#	S	$J_0(S)$
1	0	0	26	2.5	-0.048384	51	5.0	-0.177597	76	7.5	0.266339
2	0.1	0.997502	27	2.6	-0.096805	52	5.1	-0.144335	77	7.6	0.251602
3	0.2	0.990025	28	2.7	-0.142449	53	5.2	-0.110290	78	7.7	0.234559
4	0.3	0.977626	29	2.8	-0.185036	54	5.3	-0.075803	79	7.8	0.215408
5	0.4	0.960398	30	2.9	-0.224312	55	5.4	-0.041210	80	7.9	0.194362
6	0.5	0.938470	31	3.0	-0.260052	56	5.5	-0.006843	81	8.0	0.171651
7	0.6	0.912005	32	3.1	-0.292064	57	5.6	0.026971	82	8.1	0.147517
8	0.7	0.881201	33	3.2	-0.320188	58	5.7	0.059920	83	8.2	0.122215
9	0.8	0.846287	34	3.3	-0.344296	59	5.8	0.091703	84	8.3	0.096006
10	0.9	0.807524	35	3.4	-0.364295	60	5.9	0.122033	85	8.4	0.069157
11	1.0	0.765198	36	3.5	-0.380182	61	6.0	0.150645	86	8.5	0.041939
12	1.1	0.719622	37	3.6	-0.391769	62	6.1	0.177291	87	8.6	0.014623
13	1.2	0.671133	38	3.7	-0.399230	63	6.2	0.201747	88	8.7	-0.012523
14	1.3	0.620086	39	3.8	-0.402556	64	6.3	0.223812	89	8.8	-0.039234
15	1.4	0.566855	40	3.9	-0.401826	65	6.4	0.243310	90	8.9	-0.065253
16	1.5	0.511828	41	4.0	-0.397150	66	6.5	0.260094	91	9.0	-0.090333
17	1.6	0.455402	42	4.1	-0.388670	67	6.6	0.274043	92	9.1	-0.114239
18	1.7	0.397985	43	4.2	-0.376557	68	6.7	0.285065	93	9.2	-0.136748
19	1.8	0.339986	44	4.3	-0.361011	69	6.8	0.293096	94	9.3	-0.157655

20	1.9	0.281819	45	4.4	- 0.342257	70	6.9	0.298102	95	9.4	- 0.176772
21	2.0	0.223890	46	4.5	- 0.320543	71	7.0	0.300079	96	9.5	- 0.193929
22	2.1	0.166607	47	4.6	- 0.296138	72	7.1	0.299051	97	9.6	- 0.208979
23	2.2	0.110362	48	4.7	- 0.269331	73	7.2	0.295071	98	9.7	- 0.221795
24	2.3	0.055540	49	4.8	- 0.240425	74	7.3	0.288217	99	9.8	- 0.232276
25	2.4	0.002508	50	4.9	- 0.209738	75	7.4	0.278596	100	9.9	- 0.240341
-	-	-	-	-	-	-	-	-	101	10.	- 0.245936

There is also a second solution of Bessel's zero order equation containing a logarithm term. Omitting the details of calculation, we find a second solution:

$J_0(s) = \ln(s) + (s^2/2^2) - (s^4/2^2 4^2[1+(1/2)] + s^6/(2^2 4^2 6^2)[1+(1/2)+(1/3)] - s^8/(2^2 4^2 6^2 8^2)[1+(1/2)+(1/3)+(1/4)] + \ldots$ ... (37A)

The eqn. (37A) is Neumann's Bessel function of the second kind of order zero and denoted as $Y_0(s)$. Thus the general solution of Bessel's equation of order zero is: $R = C_1 J_0(s) + C_2 Y_0(s)$; $0<s<\infty$.

Case II: n = 1,2,3, ...: For this case, omitting details, which is similar to those encountered in obtaining $J_0(s)$, we get a solution: $J_n(s) = \sum_{j=0}^{\infty}[(-1)^j/\{j!(n+j)!\}](s/2)^{n+2j} =[s^n/\{(2^n)n!\}][1-\{s^2/\{2^2(n+1)\}\}+\{s^4/\{2^4(2!)(n+1)(n+2)\}\}-\{s^6/\{2^6(3!)(n+1)(n+2)(n+3)\}\}+\ldots]$.

This is the Bessel function of the first kind of order n. A second, linearly independent solution is:

$Y_n(s)=J_n(s)\ln(s)-(1/2)\sum_{j=0}^{n-1}[\{(n-j-1)!/j!\}\{(x/2)^{-n+2j}\}]-(1/2)\sum_{j=0}^{\infty}[\{(-1)^j/(j!)(n+j)!\}\{(x/2)^{n+2j}\}][\psi(j)+\psi(n+j)]$, where $\psi(m) = 1 + (1/2) + (1/3) + \ldots +(1/m)$ and $\psi(0) = 0$. This is Neumann's Bessel [Ref-5:p-377] function of the second kind of order n.

Case 1 and 2 can now be combined and substituting n=0 in $J_n(s)$ and $Y_n(s)$ of case 2 yields $J_0(s)$ and $Y_0(s)$. Thus, for n = 0,1,2,3, ..., the general solution of Bessel equation is $R = C_1 J_n(s) + C_2 Y_n(s)$.

It should be remarked here only that $J_n(s)$ and $Y_n(s)$ are oscillating functions of their arguments, that both go to zero as $s=kr \to\infty$, and that J_n is the solution which is regular at r = 0, a point for which eqn. (36) has a singularity. For real n and k, then the integrals are of the following form (kr=s):

$R(r) = A_nJ_n(kr) + B_nY_n(kr)$; $k \neq 0$,	$\Theta(\theta)=C_n\text{Cos}(n\theta) + D_n\text{Sin}(n\theta)$; $n \neq 0$,	$Z(z) = E_ke^{kz} + F_ke^{-kz}$; $k \neq 0$,
$R(r) = Ar^n + Br^{-n}$; $k = 0$,	$\Theta(\theta) = C\theta + D$; $n = 0$,	$Z(z) = E_z + F$; $k=0$.

If n and k are both zero, the solution of Laplace's equation in cylindrical coordinates becomes:

$$\Phi = [A \ln(r) + B][C\theta + D][E_z + F] \qquad \ldots(38).$$

Solutions for certain problems involving a long straight conductor or wire (but not short cylindrical segment) can be worked out easily as electrostatic potential (Φ) for these situations are independent of the coordinate z, and the Laplace's equation becomes:

$$\nabla^2\Phi = (1/r)(\partial/\partial r)[r(\partial\Phi/\partial r)] + (1/r^2)(\partial^2\Phi/\partial\theta^2) = 0 \ldots(38A)$$

Substituting $\Phi = Y(r)S(\theta)$ we get (by separating the variables)-

$$(r/Y)[(d/dr)(r(dY/dr))] = -(1/S)(d^2S/d\theta^2) = k \qquad \ldots. (38B);$$

where k is the (usual) separation constant. The θ-equation has solutions $\sin(k^{1/2}\theta)$ and $\cos(k^{1/2}\theta)$. But if these solutions are to make sense physically, each $[S(\theta)]$ must be a single-valued function of θ; thus:

$$\sin(k^{1/2}(\theta + 2\pi)) = \sin(k^{1/2}\theta), \text{ and } \cos(k^{1/2}(\theta + 2\pi)) = \cos(k^{1/2}\theta).$$

That is, after θ has gone through its full range from 0 (zero) to 2π, the function must join smoothly to its value at $\theta = 0$. This requires n as non-negative integers.

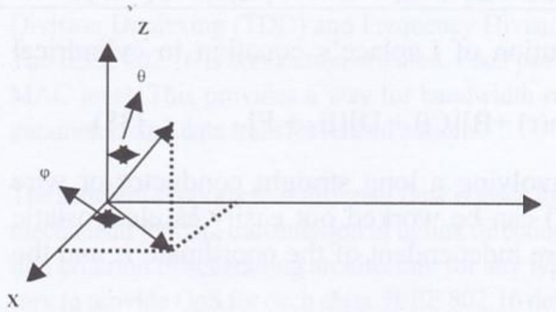
The r-equation, on the other hand, has solutions r^n or r^{-n} for non-zero n; and if $n = 0$, then either $Y(r) = \ln(r)$ or $Y(r) = $ constant. Hence, the required solutions of the Laplace's equation, the so called zonal harmonics, are:

1	$\ln(r)$	These functions form the fundamental (complete) set of solutions for the variables r, θ in cylindrical coordinates, and the potential $\Phi(r, \theta)$ may be developed as a superposition[2] of cylindrical harmonics.
$r^n\cos(n\theta)$	$r^{-n}\cos(n\theta)$	
$r^n\sin(n\theta)$	$r^{-n}\sin(n\theta)$	

(G.) Solutions to Laplace's Equation in Sperical Coordinates: The Laplace's equation in spherical coordinates [Ref-5: p-52] can be written as:

[2] This is because if Φ_1, Φ_2, …, Φ_n are all solutions of Laplace's equation, then their linear combination is also a solution. We may superimpose two or more solutions of Laplace's equation in such a way that the resulting solution satifies a given set of boundary conditions.

$\nabla^2\Phi=(1/r^2)(\partial/\partial r)[r^2(\partial\Phi/\partial r)]+[1/(r^2 Sin\theta)][(\partial/\partial\theta)Sin\theta(\partial\Phi/\partial\theta)]+[1/(r^2 Sin^2\theta)](\partial^2\Phi/\partial\varphi^2)$ ...(39),



where: r is radial direction, θ is polar angle, and φ is azimuthal angle[3].

It is intended first to find the solution for Φ as an independent of azimuthal angle, φ. Many interesting electrostatic field problems fall into this category.

Case 1: Zonal Harmonics [Φ≠Φ(φ)] [Ref-5: p-55]: If Φ≠Φ(φ), eqn. (39) becomes:

$$\nabla^2\Phi=(1/r^2)(\partial/\partial r)[r^2(\partial\Phi/\partial r)]+[1/(r^2 Sin\theta)][(\partial/\partial\theta)Sin\theta(\partial\Phi/\partial\theta)] \quad \dots (40)$$

Here, Φ=Φ(r,θ). The eqn. (40) can be solved by separation of two variables as:

$$\Phi(r,\theta) = Z(r)P(\theta) \quad \dots \quad (41)$$

Substituting eqn. (41) into eqn. (40) we get:

$$(P/r^2)\{(\partial/\partial r)[r^2(\partial Z/\partial r)]\}+[Z/(rSin\theta)][(\partial/\partial\theta)\{Sin\theta(\partial P/\partial\theta)\}] = 0$$

$$\Leftrightarrow(1/Z)[(d/dr)\{r^2(dZ/dr)\}] = -[1/\{Psin(\theta)\}](d/d\theta)\{Sin(\theta)(dp/d\theta)\} \quad \dots$$

(42)[4]

It is revealed from the above that a function of r alone is equal to a function of θ alone. It is possible if and only if, both the functions are constants. Let there exists some k∈R which satisfies the relation. The left side can be used to solve for P =P(θ):

$$[1/Sin(\theta)]\{(d/d\theta)[Sin(\theta)\{dp/d\theta\}]\}+PK = 0 \quad \dots \quad (43)$$

[3] It should be noted that r and θ have different meanings in cylindrical and spherical coordinates. In spherical coordinates r is the magnitude of the radius vector from the origin and θ is the polar angle. In cylindrical coordinates, r is the perpendicular distance from cylinder axis and θ is the azimuthal angle about this axis.

[4] Partial derivatives have been replaced by total derivatives since Z and P are each functions of one variable only: Z = Z(r), and P = P(θ).

This is the famous Legendre equation whose solutions are given by Legendre polynomials by recursion:

$$(n+1)P_{n+1}(\text{Cos}\theta) - (2n+1)(\text{Cos}\theta)P_n\text{Cos}\theta + nP_{n-1}(\text{Cos}\theta) = 0$$
$$\Leftrightarrow P_{n+1}(\text{Cos}\theta) = [(2n+1)(\text{Cos}\theta)P_n\text{Cos}\theta - nP_{n-1}(\text{Cos}\theta)]/(n+1)$$

The only solutions those physically make sense are for K = n(n+1), n is a positive integer. The solutions (of the Legendre equation) for other values of K are ill-behaved in the vicinity of θ = 0, and θ = π radians, becoming infinite or even undefined; and can not be made to fit physical boundary conditions and hence must be discarded. Thus, the solutions [Ref-5: p-376] are:

n	$P_n(\theta)$	n	$P_n(\theta)$	n	$P_n(\theta)$	n	$P_n(\theta)$
0	1	1	Cosθ	2	$(1/2)[3\text{Cos}^2\theta - 1]$	3	$(1/2)[5\text{Cos}^3\theta - 3\text{Cos}\theta]$, and so on.

We now set K = n(n+1) explicitly for Z = Z(r) and get:

$$[1/Z(r)](d/dr)[r^2\{(d/dr)Z(r)\} = n(n+1)$$

This gives two independent solutions: $Z_n = r^n, r^{-(n+1)}$. Thus, the solutions of Laplace's equation for Φ = Φ(r,θ); i.e., the zonal harmonics are:

$$\Phi = r^nP_n(\theta), \text{ or } r^{-(n+1)}P_n(\theta) \quad \ldots \quad (44)$$

where $P_n(\theta)$ is one of the polynomials listed in the above table, and n = 0, 1, 2, … . The zonal harmonics form a complete set of functions. That is, a general solution of Laplace's equation may be constructed as a superposition of these solutions[5] provided the physical problem shows the appropriate azimuthal symmetry. Several of the zonal harmonics are already known: one of the n = 0 solutions, namely electrostatic potential Φ = constant, is trivial solution of Laplace's equation, valid in any coordinate system; the zonal harmonic r^{-1} is the potential of a point charge; and $r^{-2}\cos(\theta)$ is the potential of a dipole. Let us next consider the case where Φ is dependent on φ i.e., azimuthal angle also.

Case II-Spherical Harmonics [Φ = Φ(r,θ,φ)]: A product solution in the form

$$\Phi(r,\theta,\varphi) = f_1(r)f_2(\theta)f_3(\varphi) \quad \ldots \quad (45)$$

can be assumed in which the functions f_i are real. By using equation (39) this leads to the separation:

$$(\text{Sin}^2\theta/f_1)[d/dr\{r^2(df_1/dr)\}]+\{(\text{Sin}\theta)/f_2\}\{d/d\theta[\text{Sin}\theta\{(df_2)/d\theta\}]\}+(1/f_3)[(d^2f_3)/d\varphi^2]=0 \quad \ldots(46)$$

[5] See pertinent foot-note for cylindrical harmonics given earlier.

Since the last term is a function only of φ whereas the first two terms are not, it follows that:

$$(1/f_3)[(d^2f_3)/d\varphi^2] = -m^2 \quad \ldots \quad (47)$$

in which m must be an integer if the potential is to be single-valued. Thus, $f_3(\varphi) = \{e^{\pm im\varphi}\}$; m = 0, ±1, ±2, … . Substituting eqn. (47) into eqn. (46) and division by $\mathrm{Sin}^2\theta$ gives:

$$(1/f_1)(d/dr)\{r^2(df_1/dr)\} + \{1/[f_2(\mathrm{Sin}\theta)]\}(d/d\theta)[\mathrm{Sin}\theta\{(df_2)/d\theta\}] - m^2/(\mathrm{Sin}^2\theta) = 0 \quad \ldots(48)$$

Because only the first term of the above equation is a function of r, this term must be equal to a constant k€R. Let us designate k = n(n+1) for the same reason as discussed above (i.e. for physically making sense). Thus-

$$(d/dr)[r^2(df_1/dr)] - n(n+1)f_1 = 0 \quad \ldots \quad (49), \text{ and}$$

$$(d/d\theta)[\mathrm{Sin}(\theta)(df_2/d\theta)] + [n(n+1)\mathrm{Sin}\theta - (m^2/\mathrm{Sin}\theta)]f_2 = 0 \quad \ldots(50)$$

Eqn. (49) is readily solved and gives:

$$f_1(r) = ar^n + br^{-(n+1)} \quad \ldots(51)$$

Solution of eqn. (50) is facilitated by making substitution u = Cosθ which gives:

$$(1 - u^2)(d^2f_2/du^2) - 2u(df_2/du) + [n(n+1) - \{m^2/(1-u^2)\}]f_2 = 0 \quad \ldots. (52)$$

The functions which satisfy this equation are once again the Legendre functions and two independent solutions are normally designated $P^m{}_n(u)$ and $Q^m{}_n(u)$. The later has singularities at the poles θ = 0, π and must be excluded if the polar axis is part of the region of interest. Appendix includes a discussion of the manner in which eqn. (52) is solved, together with a development of the major properties of the function $P^m{}_n(u)$, and only the principal results will be stated here.

The assumption of a power series solution of eqn. (52) for the case m=0 leads to the conclusion that, if n is an integer, $f_2(u)$ can be expressed as a polynomial which is well-behaved in the entire region $-1 \leq u \leq 1$ and given by-

$$P_n(u) = [1/(2^n n!)][(d^n/du^n)(u^2-1)^n \quad \ldots \quad (53)$$

The first few of these are given earlier[6]. If all the positive integral values of n are included, Legendre polynomials generated by eqn. (53) constitute a

[6] See between equations (43) and (44).

complete orthogonal set in the interval [-1, 1] and for this reason, non-integral values of n will not be considered.

For $m \neq 0$, the associated Legendre function $P^m{}_n$ satisfies eqn. (52), is given by-

$P^m{}_n(u) = (1-u^2)^{m/2}[d^m P_n(u)/du^m] = (\{(1-u^2)^{m/2}\}/(2^n n!))[d^{m+n}\{(u^2-1)^n\}/du^{n+m}]$ … (54)

Since $P_n(u)$ is an nth-order polynomial, m can not exceed n in value. A variety of recurrence formulas connecting associated Legendre functions and/or their derivatives for different values of the indices is given in Appendix, together with a list of the specific functions generated from eqn. (54) for low values of m & n.

The associated Legendre functions are also orthogonal in [-1,1], the normalization integral being-

$$\int_{-1}^{1} P_n{}^m(u)P_l{}^m(u)du = [\{2(n+m)\}!/\{(2n+1)(n-m)\}!]\delta_{lm} \quad \dots \quad (55)$$

When eqn. (45) is expanded in terms of the solutions which have been found for constituent functions, we get-

$$\Phi(r,\theta,\varphi) = \sum_{m=0}^{n}\sum_{n=0}^{\infty}[a_n r^n + b_n r^{-(n+1)}]p_n{}^m \text{Cos}\theta[C_m \text{Cos}(m\varphi) + d_m \text{Sin}(m\varphi)] \quad \dots(56)$$

The combination $p_n{}^m\text{Cos}\theta[C_m\text{Cos}(m\varphi)+d_m\text{Sin}(m\varphi)]$ is called a spherical harmonic. Being orthogonal in both $\text{Cos}\theta$ and φ, it is suitable for the expansion of the arbitrary functions of θ and φ in spherical coordinates in exactly the same way that a double Fourier series is used in two dimensions in rectangular coordinates.

APPENDIX

The Associated Legendre Equation: The solution of the differential equation [Ref-2: p-281, 325; Ref-5: p-57, 376],

$$(1-u^2)(d^2f_2/du^2) - 2u(df_2/du) + [n(n+1) - \{m^2/(1-u^2)\}]f_2 = 0 \quad \dots \quad (A.1)$$

will be undertaken here. As a first step, we consider the case m = 0 which results in the ordinary Legendre equation-

$$(1-u^2)(d^2g/du^2) - 2u(dg/du) + n(n+1)g = 0 \quad \dots(A.2)$$

Let a solution to (A.2) be assumed in the form,

$$g = \sum_{p=0}^{\infty}[a_p u^{s+p}] \quad \dots (A.3)$$

in which s is a constant. Then, $dg/du = \sum_{p=0}^{\infty}[(s+p)a_p u^{s+p-1}]$ and $d^2g/du^2 = \sum_{p=0}^{\infty}[(s+p)(s+p-1)a_p u^{s+p-2}]$; and substitution of these terms in (A.2) gives:

$\sum_{p=0}^{\infty}[(s+p)(s+p-1)a_p u^{s+p-2}] - \sum_{p=2}^{\infty}[(s+p-2)(s+p-3)a_{p-2}u^{s+p-2}] - 2\sum_{p=2}^{\infty}[(s+p-2)a_{p-2}u^{s+p-2}] + n(n+1)\sum_{p=2}^{\infty}[a_{p-2}u^{s+p-2}] = 0$

Since this result is to hold for all values of u, the co-efficient of each power of u must separately equal zero and, therefore,

$s(s-1)a_0 = 0$	$(s+1)sa_1 = 0$	$(s+p)(s+p-1)a_p = [(s+p-2)(s+p-1) - n(n+1)]a_{p-2}$

If s = 0, the first two of these conditions are satisfied and the third condition becomes the recursion formula:

$$a_p = -\{[(n-p+2)(n+p-1)]/p(p-1)\}a_{p-2} \quad \ldots(A.4)$$

The solution to (A.2) can then be written-

$$g = a_0[1 - \{n(n+1)/2!\}u^2 + \{n(n-2)(n+1)(n+3)/4!\}u^4 - \ldots] + a_1[u - \{(n-1)(n+2)/3!\}u^3 + \{(n-1)(n-3)(n+2)(n+4)/5!\}u^5 - \ldots] \quad \ldots \quad (A.5)$$

For non-integral n both of the series in (A.5) converge except at u = ±1. Since one series is odd and the other even, they represent linearly independent solutions of (A.2) so that (A.5) is a general solution provided that $|u|<1$. Nothing more is added by choosing s = ±1 since each choice leads to one of the series in (A.5).

If n is an even integer, it is clear that the first series in (A.5) terminates and is thus a polynomial, whereas if n is an odd integer the second series reduces to a polynomial. If the arbitrary constants a_0 and a_1 are adjusted so as to give these polynomials the value unity when u = 1, the Legendre polynomials are obtained, the first few of which are:

$$P_0(u) = 1,\ P_1(u) = u = \cos\theta,\ P_2(u) = (3/2)u^2 - 1/2 = (3/4)\cos 2\theta + (1/4),$$
$$P_3(u) = (5/2)u^3 - (3/2)u = (5/8)\cos 3\theta + (3/8)\cos\theta, \text{ etc.}$$

These polynomials can also be generated from Rodrigues' formula-

$$P_n(u) = [1/\{2^n(n!)\}](d^n/du^n)\{(u^2 - 1)^n\} \quad \ldots \quad (A.6)$$

which may be verified by expansion.

For n an integer, the non-terminating series in (A.5), with the constant suitably adjusted, is known as the Legendre function of the second kind, $Q_n(u)$. These functions are characterized by singularities at u = ±1 and must be excluded from the solutions of physical problems in regions containing the polar axis. They will not be considered here.

The Legendre polynomials $P_n(u)$ defined above satisfy (A.2) which may be written as:

$$(1-u^2)(d^2P_n/du^2) - 2u(dP_n/du) + n(n+1)P_n = 0 \ldots \quad (A.7)$$

If this equation is differentiated m times with respect to u, one obtains:

$(1-u^2)(d^2h/du^2) - 2(m+1)u(dh/du) + [n(n+1) - m(m+1)]h = 0$ ...(A.8)

in which $h(u) = d^mP_n/du^m$.

When one lets $h(u) = (1-u^2)^{-m/2}f_2(u)$, eqn. (A.8) transforms into (A.1). Thus,

$$f_2(u) = P_n^m(u) = (1-u^2)^{m/2}[d^mP_n(u)/du^m] \quad ... \quad (A.9)$$

is a solution to the associated Legendre eqn. (A.1). The functions

$$P_n^m(u) = [(1-u^2)^{m/2}/(2^n n!)][d^{n+m}/du^{n+m})(u^2-1)^n] \quad ... \quad (A.10)$$

are known as the associated Legendre functions of the first kind. Since P_n is a polynomial of order n, it follows that $P_n^m(u) = 0$ for m>n.

It is obvious that the functions $P_n^0(u)$ are identical with the polynomials $P_n(u)$ previously listed. If one used (A.9) [Ref-5: p-376],

$P_1^1(u) = (1-u^2)^{1/2} = \text{Sin}\theta$	$P_2^2(u) = 3(1-u^2) = (3/2) - (3/2)\text{Cos}2\theta$
$P_2^1(u) = 3u(1-u^2)^{1/2} = (3/2)\text{Sin}2\theta$	$P_3^2(u) = 15u(1-u^2) = (15/4)\text{Cos}\theta - (15/4)\text{Cos}3\theta$
$P_3^1(u) = (3/2)(5u^2-1)(1-u^2)^{1/2} = (3/8)\text{Sin}\theta + (15/8)\text{Sin}3\theta$	$P_3^3(u) = 15(1-u^2)^{3/2} = (45/4)\text{Sin}\theta - (15/4)\text{Sin}3\theta$

A second generating function for the Legendre polynomials is given by the expression: $f(u,t) = [1-(2ut-t^2)]^{-1/2}$ which can be expanded into the series:

$f(u,t) = 1 + [(1/2)/1!](2ut - t^2) + [(1/2)(3/2)/2!](2ut - t^2)^2 + ... + [[(1/2)(3/2)... \{(2n-1)/2\}]/(n!)](2ut-t^2)^n + ...$

If this is re-arranged as a power series in t one obtains:

$$f(u,t) = 1 + ut + [\{(3u^2-1)/2\}t^2] + [\{(5u^2-3u)/2\}t^3] + ...$$

and the co-efficients of the different powers of t are recognized to be the Legendre polynomials, so that-

$$(1-2ut+t^2)^{-1/2} = \Sigma_{n=0}^{\infty}t^nP_n(u) \quad ...(A.11)$$

Differentiation of (A.11) with respect to t gives-

$$(u-t)/[(1-2ut+t^2)^{3/2}] = \Sigma_{n=0}^{\infty}nt^{n-1}P_n(u) \quad ...(A.12)$$

which can be written as: $(u-t)\Sigma_{n=0}^{\infty}t^nP_n(u) = (1-2ut+t^2)\Sigma_{n=0}^{\infty}nt^{n-1}P_n(u)$.

Equating co-efficients of t^n, one determines that-

$$(n+1)P_{n+1}(u) - (2n+1)uP_n(u) + nP_{n-1}(u) = 0 \quad ... \quad (A.13)$$

This recurrence relation will permit the determination of any Legendre polynomial if two successive ones are known. Differentiation of (A.11) with respect to u yields-

$$t/[(1-2ut+t^2)^{3/2}] = \Sigma_{n=0}^{\infty} t^n P_n'(u) \quad \ldots(A.14)$$

which can be re-arranged as: $t\Sigma_{n=0}^{\infty} t^n P_n(u) = (1-2ut+t^2)\Sigma_{n=0}^{\infty} t^n P_n'(u)$. The co-efficients of t^n gives-

$$P_{n-1}(u) = P_n'(u) - 2uP_{n-1}'(u) + P_{n-2}'(u) \quad \ldots \quad (A.15)$$

Knowledge of the derivative of two successive Legendre polynomials will thus permit determination of any other through the use of (A.15).

Alternatively, (A.14) can be re-arranged with the aid of (A.12) to give: $t\Sigma_{n=0}^{\infty} nt^{n-1} P_n(u) = (u-t)\Sigma_{n=0}^{\infty} t^n P_n'(u)$ which yields the recursion formula-

$$nP_n(u) = uP_n'(u) - P_{n-1}'(u) \quad \ldots(A.16)$$

from which the derivative of any Legendre polynomial can be determined if one polynomial and its derivative are known.

Combination of (A.15) and (A.16) delivers the useful differentiation formula-

$$(1-u^2)(dP_n/du) = nP_{n-1}(u) - nuP_n(u) \quad \ldots \quad (A.17)$$

Recurrence relations for the associated Legendre [Ref-5: p-376] functions follow readily with the aid of (A.10). Two of the more important formulas are-

$$(n-m+1)P_{n+1}^m - (2n+1)uP_n^m + (n+m)P_{n-1}^m = 0 \quad \ldots \quad (A.18)$$

$$(1-u^2)(dP_n^m/du) = (n+m)P_{n-1}^m - nuP_n^m \quad \ldots \quad (A.19)$$

One of the most useful properties of the Legendre polynomials is their orthogonality in the interval $-1 \leq u \leq 1$. This can be established by returning to the differential equation (A.7). The two polynomials $P_l(u)$ and $P_n(u)$ satisfy this equation in the forms-

$$(d/du)[(1-u^2)P_l'(u)] + l(l+1)P_l(u) = 0 \quad \ldots \quad (A.20)$$

$$(d/du)[(1-u^2)P_n'(u)] + n(n+1)P_n(u) = 0 \quad \ldots \quad (A.21)$$

Upon multiplying (A.20) and (A.21) by $P_n(u)$ and $P_l(u)$ respectively, subtracting, and then integrating from -1 to $+1$, one obtains-

$$(l-n)(l+n+1)\int_{-1}^{1} P_l(u)P_n(u)du = [\{(1-u^2)[P_n(u)P_l'(u) - P_l(u)P_n'(u)]\}]_{-1}^{+1} = 0 \quad \ldots \quad (A.22)$$

in which the right side of (D.22) has been achieved through integration by parts. Therefore,

$$\int_{-1}^{1} P_l(u)P_n(u)du = 0, 1 \neq 0 \quad \ldots \quad (A.23)$$

and the Legendre polynomials are orthogonal.

To determine the value of this integral if $l = n$, the generating function (D.11) can be used. Squaring both sides and integrating with respect to u gives:

$\int_{-1}^{1}(1-2ut+t^2)^{-1}du = \int_{-1}^{1}[P_0(u) + tP_1(u) + \ldots + t^nP_n(u) + \ldots]^2du$, which becomes:

$$[(-1/2t)\{\ln(1-2ut+t^2)\}]_{-1}^{1} = \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}P_n^2(u)du,$$

with the reduction of the right side occurring by virtue of (A.23). Insertion of the limits yields:

$$(1/t)[\ln\{(1+t)/1-t)\}] = 2\sum_{n=0}^{\infty}[t^{2n}/(2n+1)] = \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}P_n^2(u)du,$$

in which the logarithmic function has been replaced by its series expansion. Equating coefficients of like powers of t, one obtains-

$$\int_{-1}^{1}P_n^2(u)du = [2/(2n+1)] \quad \ldots(A.24)$$

The associated Legendre functions P_l^m and P_n^m, which satisfy-

$$(d/du)[(1-u^2)(dP_l^m/du)] + [l(l+1) - \{m^2/(1-u^2)\}]P_l^m = 0 \quad \ldots(A.25)$$

$$(d/du)[(1-u^2)(dP_n^m/du)] + [n(n+1) - \{m^2/(1-u^2)\}]P_n^m = 0 \quad \ldots(A.26)$$

are also orthogonal in the same interval. This can be established by a repetition of the foregoing procedure. If (A.25) and (A.26) are multiplied by P_n^m and P_l^m respectively, the difference taken, and the result integrated, the result is that-

$$\int_{-1}^{1}P_l^m(u)P_n^m(u)du = 0,\ l \neq n \quad \ldots(A.27)$$

The normalization integral is:

$\int_{-1}^{1}[P_n^m(u)]^2du = \int_{-1}^{1}(1-u^2)^m(d^mP_n/du^m)(d^mP_n/du^n)du$, which reduces to-

$$\int_{-1}^{1}[P_n^m(u)]^2du = -\int_{-1}^{1}(d^{m-1}P_n/du^{m-1})[(d/du)\{(1-u^2)^m(d^mP_n/du^m)\}]du \quad \ldots(A.28)$$

after integration by parts. If in (A.8) one replaces m by m-1 and multiplies through by $(1-u^2)^{m-1}$, there results:

$$(d/du)[(1-u^2)^m(d^mp_n/du_m)] = -(n-m+1)(n+m)(1-u^2)^{m-1}(d^{m-1}P_n/du^{m-1}).$$

Substitution of this expression in (A.28) gives-

$$\int_{-1}^{1}[P_n^m(u)]^2du = (n-m+1)(n+m)\int_{-1}^{1}(1-u^2)^{m-1}(d^{m-1}P_n/du^{m-1})(d^{m-1}P_n/du^{m-1})du$$
$$= (n+m)(n-m+1)\int_{-1}^{1}[P_n^{m-1}(u)]^2du \quad \ldots(A.29)$$

with the aid of (A.28). Use of the reduction formula (A.29) yields:

$$\int_{-1}^{1}[P_n^m(u)]^2du = [(n+m)!/(n-m)!]\int_{-1}^{1}[P_n^0(u)]^2du.$$

Finally, through the use of (A.24), we get-

$\int_{-1}^{1} P_n^m(u)\, P_l^m(u)du = [(n+m)!/\{(2n+1)(n-m)\}]\delta_{ln}$ (A.30)

This result is of considerable importance since it provides the opportunity to expand a function $f_2(u)$ in terms of associated Legendre polynomials with the coefficients individually determinable from (A.30). This technique greatly facilitates the solution of many boundary value problems.

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